CASIMIR OPERATORS OF LIE ALGEBRAS WITH A NILPOTENT RADICAL

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Résumé. We show that a Lie algebra having a nilpotent radical has a fundamental set of invariants consisting of Casimir operators. We give a different proof of this fact in the special and well-known case where the radical is abelian.

1. Introduction

An important problem arising in the representation theory of Lie groups is the determination of the invariants of a given representation. If ρ is a representation of the Lie group G in a finite dimensional vector space V, the invariants of ρ are the elements v of V for which the equality $\rho(g)v = v$ holds for all $g \in G$. A map ρ is a representation of a connected Lie group G in the finite dimensional vector space V if and only if its differential $d\rho$ is a representation of the Lie algebra L of G in V. Moreover, for any $v \in V$ we have $\rho(G)v = v$ if and only if $d\rho(L)v = 0$, and the latter condition defines v as an invariant of $d\rho$. The invariants of G and L are therefore the same and in general they are more easily analyzed on Lie algebras.

When ρ is the adjoint representation Ad of G in its Lie algebra L, the invariants of the corresponding representation of G in the symmetric algebra $S(L)$ are called the polynomial invariants of L. The algebra S^I generated in $S(L)$ by the invariant polynomial functions is algebraically isomorphic to the algebra of Casimir operators [5], which is the center $\mathcal{Z}(L)$ of the universal enveloping algebra $\mathfrak{A}(L)$ of L. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of the finite n-dimensional Lie algebra L, with structure constants C_{ij}^k in this basis, it is well known [7] that the infinitesimal generators $\tilde{v}_i$ of the adjoint action of G on $S(L)$ are operators of the form

$$\tilde{v}_i = \sum_j \sum_k C_{ij}^k x_k \partial_j,$$

where $\{x_1, \dots, x_n\}$ is a coordinate system associated with the basis $\mathcal{B}$, and an element $p \in S(L)$ is in S^I if and only if

$$\tilde{v}_i \cdot p \;=\; 0, \qquad \text{for all } i = 1, \dots, n. \tag{1.1}$$

This shows that the polynomial invariants of L are completely determined by its structure constants. A maximal set of functionally independent solutions to (1.1) for all possible types of functions p is referred to as a fundamental set of invariants of L. Thus isomorphic Lie algebras have the same fundamental sets of invariants. We shall assume that the base field $\mathbb{K}$ of L is of characteristic zero.

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2. Characterization of the invariants

We write the Levi decomposition of a given finite dimensional Lie algebra L in the form $L = \mathcal{S} \dot{+} \mathcal{R}$, where $\mathcal{S}$ is the Levi factor and the ideal $\mathcal{R}$ is the radical of L.

Lemma 1. *If the Lie algebra L has a nilpotent radical, then it is isomorphic to an algebraic Lie algebra.*

Démonstration. By Ado's Theorem, L has a faithful representation ϕ in a finite-dimensional vector space V, in which elements in the nilradical of L are represented by nilpotent endomorphisms [8]. Since the radical $\mathcal{R}$ of L is nilpotent, it is equal to its nilradical, and the Lie algebra $\phi(\mathcal{R})$ which consists of nilpotent endomorphisms and is consequently algebraic [3]. Moreover, $\phi(\mathcal{R})$ is the radical of $\phi(L)$, and a subalgebra of $\mathfrak{gl}(V)$ is algebraic if and only if its radical is algebraic [3]. Consequently, $\phi(L)$ is algebraic. $\square$

All semisimple and nilpotent Lie algebras belong to the class of Lie algebras with a nilpotent radical. This is also the case for all perfect Lie algebras, and more generally for derived subalgebras of finite dimensional Lie algebras, precisely because the radical of such Lie algebras is nilpotent [8].

By a result of [1], every perfect Lie algebra, i.e. a Lie algebras L for which $[L, L] = L$, has a fundamental set consisting of polynomial invariants. However, we notice that this property also holds for Lie algebras with an abelian radical. Indeed, write the Levi decomposition of L in the form

$$L = \mathcal{S} \oplus_\pi \mathcal{R}, \tag{2.1}$$

where π is the restriction to the semisimple Lie algebra $\mathcal{S}$ of the adjoint representation of L in the radical $\mathcal{R}$. If $\mathcal{R}^{\mathcal{S}} = \{v \in \mathcal{R} \colon \pi(\mathcal{S})v = 0\}$ is the set of invariants of this representation, then because π is semisimple, we have the direct sum of vector space $\mathcal{R} = \mathcal{R}^{\mathcal{S}} \dot{+} [\mathcal{S}, \mathcal{R}]$.

Theorem 1. *Let $L = \mathcal{S} \oplus_\pi \mathcal{R}$ be a Levi decomposition of L.*

- (a): *The Lie algebra L is perfect if and only if $\mathcal{R}^{\mathcal{S}} \subseteq [\mathcal{R}, \mathcal{R}]$.*
- (b): *If the radical $\mathcal{R}$ of L is abelian, then L has a fundamental set of invariants consisting of polynomial functions.*

Démonstration. We know that L is perfect if and only if $\mathcal{R} = [\mathcal{S}, \mathcal{R}] + [\mathcal{R}, \mathcal{R}]$. Writing the right hand side of this last equality as a direct sum $[\mathcal{S}, \mathcal{R}] \dot{+} W \cap [\mathcal{R}, \mathcal{R}]$ of vector subspaces, where W is a complement subspace of $[\mathcal{S}, \mathcal{R}]$ in $\mathcal{R}$, we see that L is perfect if and only if $\mathcal{R}^{\mathcal{S}} \subseteq [\mathcal{R}, \mathcal{R}]$, which proves part (a). For part (b) we note first that by a result of [9], if the representation π does not possess a copy of the trivial representation then L is perfect, and the result follows. If π does have a copy of the trivial representation, then $\mathcal{R}^{\mathcal{S}} \neq 0$, and by part (a) above, L is not perfect. However, L is in this case a direct sum of the perfect ideal $\mathcal{S} \dot{+} [\mathcal{S}, \mathcal{R}]$ and the abelian ideal $\mathcal{R}^{\mathcal{S}}$. It then follows again that all the invariants of L can be chosen to be polynomials. $\square$

Lemma 2. *A Lie algebra with a nilpotent radical has a fundamental set of invariants consisting of rational functions.*

Démonstration. Lemma 1 reduces the proof to the case of algebraic Lie algebras, and the lemma readily follows from a result of J. Dixmier [4] asserting that any algebraic Lie algebra has a fundamental set of invariants consisting of rational functions.

□

In the sequel we shall denote by $\mathrm{Fract}(A)$ the field of fractions of a Noetherian and integral ring A. Set $\mathfrak{K}(L) = \mathrm{Fract}(\mathfrak{A}(L))$, and $\mathsf{K}(L) = \mathrm{Fract}(S(L))$, and for each $x \in L$, denote by $\mathrm{ad}_{\mathsf{K}(L)}\, x$ the derivation of $\mathsf{K}(L)$ that extends $\mathrm{ad}_L\, x$ and thus defines a representation of L in $\mathsf{K}(L)$. The invariants of this representation are called the rational invariants of L. It should be noted that $\mathsf{K}(L)$ is isomorphic the field $\mathbb{K}(x_1, \ldots, x_n)$ of rational functions in n commuting variables. Similarly, for each $x \in L$, denote by $\mathrm{ad}_{\mathfrak{K}(L)}\, x$ the derivation of $\mathfrak{K}(L)$ that extends $\mathrm{ad}_L\, x$. Finally, denote by $\mathcal{C}(L)$ and $\mathcal{F}(L)$ the center of $\mathfrak{K}(L)$ and $\mathsf{K}(L)$ respectively, when they are endowed with the adjoint representation. The center S^I of $S(L)$ is also given by $S^I = \left\{p \in S(L)\colon\ \mathrm{ad}_{S(L)}(L)(p) = 0\right\}$, and we have $\mathrm{Fract}(S^I) \subset \mathcal{F}(L)$. By a result of [10], $\mathcal{C}(L)$ and $\mathcal{F}(L)$ are isomorphic fields. Moreover, we have the following result [1] in which L^* denote the dual vector space of L.

Lemma 3. *We have $f \in \mathcal{F}(L)$ if and only if there exists some weight $\lambda \in L^*$ of the adjoint representation of L in $S(L)$ such that $f = p_1/p_2$, where $p_1, p_2 \in S(L)_\lambda = \left\{p \in S(L)\colon\ \mathrm{ad}_{S(L)}\, x\,(p) = \lambda(x)p,\ \text{for all } x \in L\right\}$*

If λ is in L^*, an element of $S(L)_\lambda$ is called a λ-semi-invariant of L in $S(L)$. It is clear that if the weight space of any such λ is not reduced to zero, then λ defines a one dimensional representation of L in $\mathbb{K}$, which vanishes on any perfect subalgebra of L.

Theorem 2. *If the radical of the Lie algebra L is nilpotent, then L has a fundamental set of invariants consisting of Casimir operators.*

Démonstration. Since L has a nilpotent radical, we may assume by Lemma 1 that it is algebraic. It has therefore a fundamental set of invariants that consists of rational invariants, by Lemma 2. In this case, we readily see that a sufficient condition for L to have only polynomial invariants is for the equality $\mathcal{F}(L) = \mathrm{Fract}(S^I)$ to hold, and because of Lemma 3, to prove this last equality it suffices to verify that the only weight of the adjoint representation of L in $S(L)$ is 0. Consider the Levi decomposition $L = \mathcal{S} \dot{+} \mathcal{R}$, and let λ be any weight of $\mathrm{ad}_{S(L)}$. If $x \in \mathcal{S}$, then clearly $\lambda(x) = 0$. On the other hand if $x \in \mathcal{R}$, $\mathrm{ad}_L\, x$ is nilpotent and $\mathrm{ad}_{S(L)}\, x$ is locally nilpotent, and hence $\lambda(x) = 0$. The rest of the theorem follows from the isomorphism between $\mathcal{F}(L)$ and $\mathcal{C}(L)$. □

Special cases of Theorem 2 are known for L semisimple [6], L nilpotent [2], and L perfect [1]. This theorem therefore unifies and extends seemingly unrelated results asserting that the invariants for these particular types of Lie algebras can all be chosen as Casimir operators, and shows that the only Lie algebras that may not have a fundamental set of invariants consisting of Casimir operators are to be found only among the non nilpotent solvable Lie algebras.

Références

[1] L. Abellanas, L. Alonso Martinez, A general setting for Casimir Invariants, J. Math. Phys. 16 (1975) 1580-1584.

[2] P. Bernat, Sur le Corps des quotients de l'algèbre enveloppante d'une algèbre de Lie, C. R. Acad. Sci. Paris. 254 (1962) 1712-1714.

[3] C. Chevalley, Théory des groupes de Lie III, Hermann, Paris, 1955.

[4] J. Dixmier, Sur les représentations unitaires des groupes de Lie nilpotents, Bull. Soc. Math. France 85 (1957) 325-388.

[5] J. Dixmier, Sur l' algèbre enveloppante d'une algèbre de Lie nilpotente, Arch. Math. 10 (1959) 321-326.

[6] J. Dixmier, Enveloping algebras, Gauthier-Villars, Paris, 1974.

[7] A. T. Fomenko, V.V. Trofimov, Integrable systems on Lie algebras and symmetric spaces, Gordon & Breach, New York, 1988.

[8] N. Jacobson, Lie algebras, Dover, New York, 1962.

[9] J.-C. Ndogmo, Invariants of a semi-direct sum of Lie algebras, J. Phys. A : Math. Gen. 37 (2004) 5635-5647.

[10] R. Rentschler, Sur le centre du corps enveloppant d'une algèbre de Lie résoluble. C. R. Acad. Sci. Paris (A) 276 (1973) 21-24.

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